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A stochastic model of a pension fund

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BELLMAN’S PRINCIPLE OF OPTIMALITY

- An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (Bellman, 1957)

APPLICATIONS IN ECONOMICS

- The first known application of a Bellman equation in economics is due to Martin Beckmann and Richard Muth. Martin Beckmann also wrote extensively on consumption theory using the Bellman equation in 1959. His work influenced Edmund S. Phelps, among others.
- A celebrated economic application of a Bellman equation is Merton's seminal 1973 article on the intertemporal capital asset pricing model.

BASIC FEATURES OF MODEL

- Following Emms and Haberman (2008) we consider a pension fund at the size X(s) at a fixed time of retirement s. An annuity bought at that time makes the pensioner receive a constant cash flow b_s for each time unit for life.
- As an alternative the pensioner can defer the purchase of an annuity and invest the fund X(s) at he risk free rate in a single asset providing a rate of return r. We assume that the pensioner withdraws b_s per time unit until time T at which point the pensioner is required to purchase an annuity with the remainder of the fund.

A SIMPLE DETERMINISTIC MODEL

- Thus, if F(t) denotes the size of the fund at time t the dynamics of the fund can be written as

$$\frac{dF}{dt} = rF - b_s$$

- This is a standard linear differential equation of the first degree that along with the boundary condition $F(s) = X(s)$ has the solution

$$F(t) = \frac{b_s}{r} + \left(X(s) - \frac{b_s}{r} \right) e^{r(t-s)}$$

A NATURAL INTERPRETATION

- If at each time less income is drawn from the fund than its return over the unit of time, i.e., if $b_s < rX(s)$, the fund grows at an exponential rate and the pension fund outperforms the annuity.
- If on the other hand $b_s > rX(s)$ there is a time t^* at which $F(t^*) = 0$ and the fund is exhausted.
- It has to be assumed that $t^* > T$ for otherwise the annuity is preferable to income drawdown.

EXTENSION OF MODEL TO COVER DIFFERENT INVESTMENT STRATEGIES

- This model can be extended to cover different investment strategies.
- Let $w(t)$ denote the proportion of the pension fund invested in a risky asset at time $t > s$ with the other part of the fund invested in the risk free asset.
- In finance, in particular the Black–Scholes model, changes in the logarithm of exchange rates, price indices, and stock market indices are assumed normal.
- Hence, we assume that the price of the risky asset is lognormally distributed with constant drift λ and constant volatility σ. Given the long-term horizon of a pension fund we do not consider the normalcy assumption overly restrictive.

ASSETS OF THE FUND

- Thus, the fund is composed of two assets, i.e., the risk-free bond B and the risky asset S.
- The dynamics of the two assets are as follows:
- $dB(t) = rB(t)dt$
- $dS(t) = \lambda S(t)dt + \sigma S(t)dW(t)$
- W(t) is the usual Wiener process.
- In finance this is essentially known as the Black-Scholes model.
- We restate that at any time the proportion w(t) is invested in the risky asset.

A STOCHASTIC DIFFERENTIAL EQUATION

- Combining the equations above we obtain a stochastic differential equation for the change in the value of the fund is

$$dX(t) = \left[X(t)r\left(1 - w(t)\right) + \mathrm{X(t)\lambda w(t)} - \mathrm{b}\left(\mathrm{t, X(t)}\right)\right]dt + X(t)w(t)\mathrm{\sigma dW(t)}$$

$$= \left[X(t)\mathrm{(w(t)(\lambda - r) + r)} - \mathrm{b}\left(\mathrm{t, X(t)}\right)\right]dt + X(t)w(t)\mathrm{\sigma dW(t)}$$

- where the pension benefits b(t, X(t)) can depend on time and the current state of the fund.

THE RISK-NEUTRAL MEASURE

- We denote the "objective" probability measure that governs the Black-Scholes model above by the letter P. We say that the P-dynamics of the S-process is that of

$$dS(t) = \lambda S(t)dt + \sigma S(t)dW(t)$$

- We now define another probability measure Q under which the S-process has a different probability distribution so that the Q-dynamics of S become

$$dS(t) = rS(t)dt + \sigma S(t)dW^{Q}(t)$$

- where W^{Q} is is a Q-Wiener process. This Q-measure is sometimes called the risk adjusted measure. Björk (2009) states that this measure most often is called the martingale measure, the reason being that under Q the normalized process $\frac{S(t)}{B(t)}$ turns out to be a Q-martingale.

EXISTENCE OF A UNIQUE MARTINGALE MEASURE Q

- Loosely speaking, the market is arbitrage free if and only there exists a martingale measure Q equivalent to the physical measure P.
- The measure Q is unique if we add the assumption that the market is complete in the sense that any claim can be hedged.
- These two statements are termed the first and second fundamental theorems of the martingale approach to arbitrage theory. Björk (2009). Needless to say, in the formal derivation of the theory, the terms arbitrage and completeness are given precise mathematical definitions.
- Essentially, there exists a Radon-Nikodym derivative $f = \frac{dQ}{dP}$ that transforms the objective P-measure into a martingale measure Q.

PERFORMANCE ASSESSMENT OF THE FUND

- The performance of the fund is assessed on the basis of a relative comparison with the benchmark fund F(t) by

$$Z(t) = \frac{X(t)}{F(t)}$$

  with $Z(s) = 1$ at time $t = s$.
- As X differs from F in having some of the fund invested in equity Z measures the benefit of such an asset allocation compared to investing solely in the risk-free asset.

MINIMIZATION OF LOSS

- In order to determine the optimal asset allocation w(t) we minimize the expected total discounted loss measured by $L = L(t, Z(t))$ over the planning horizon

$$\mathbb{E}_{s,x}\left[\int_s^T e^{-\rho(u-s)}\, L\big(u, Z(u)\big)du + \epsilon e^{-\rho(T-s)} L\big(T, Z(T)\big)\right]$$

- where ρ is the subjective discount rate and ϵ is a measure of any terminal cost at compulsory retirement.
- Emms and Haberman choose the function L is so that there is no loss if the current pension fund is equal to the pension fund, i.e., $\mathrm{L} = 0$ at $Z = 1$. Unlike them we see no need to normalize the loss function in this manner.
- We postulate that $L' < 0$ and $L'' > 0$, i.e., that L is a decreasing convex function; with convexity reflecting risk aversion.

STOCHASTIC OPTIMAL CONTROL THEORY

- The problem is one of minimization in a stochastic framework and lends itself to stochastic optimal control theory. We want to minimize the value function V(t,x) and the problem becomes

$$\mathrm{V}(\mathrm{t},\mathrm{x}) = \min_{w} \mathbb{E}_{s,x}\left[\int_t^T e^{-\rho(u-s)} L\big(u, Z(u)\big) du + \epsilon e^{-\rho(T-s)} L\big(T, Z(T)\big)\right]$$

We assume that V is twice differentiable.

APPLICATION OF HJB AND ITO'S LEMMA

- By a standard application of Bellman's principle of optimality and expanding V by Ito's lemma we obtain the Hamilton-Jacobi-Bellman (HJB) equation

$$V_t + \min_w \{(x(w(\lambda - r) + r - b(t,x))V_x + \frac{1}{2}(xw\sigma)^2 V_{xx}\} + e^{-\rho(T-s)}L(T,z) = 0$$

- with terminal boundary conditon

$$V\big(T, X(t)\big) = \epsilon e^{-\rho(T-s)} L\big(T, Z(T)\big)$$

- We note that the randomness has disappeared and now we are operating in a deterministic framework.

FIRST AND SECOND ORDER CONDITIONS

- The first order condition obtained by differentiating the HJB equation with respect to w is

$$w = -\frac{\beta V_x}{\sigma x V_{xx}}$$

- where β is the Sharpe ratio

$$\beta = \frac{\lambda - r}{\sigma}.$$

- The second order condition for a local minimum is $V_{xx} > 0$.

A DIFFICULT EQUATION

- Substituting the first order condition into the HJB equation gives

$$V_t + \big(rx - b(t,x)\big)V_x - \frac{1}{2}\beta^2 \frac{V_x^2}{V_{xx}} + e^{-\rho(t-s)}L(t,z) = 0$$

- Solving this equation is made more difficult by the presence of the first order term $\big(rx - b(t,x)\big)V_x$.
- As pointed out by Emms and Haberman if b is proportional to x then a power function for V would yield an analytical solution. In general, however, the value function V depends on x and t and must be calculated numerically.

AVENUES TOWARD A SOLUTION

- One possible avenue is to consider loss functions depending solely on the pension fund performance, i. e. $L = L(z)$, and rewrite the HJB equation so that $V = V(t, z)$.
- By way of the chain rule of differentiation this change of variables results in that the relevant partial derivatives change as follows:

V_t becomes $V_t - \frac{zV_z}{F}\frac{dF}{dt}$

V_x becomes $\frac{V_z}{F}$

V_{xx} becomes $\frac{V_{zz}}{F^2}$

A SIMPLIFIED HJB EQUATION

- By applying the basic dynamic equation of the fund the HJB equation becomes

$$V_t + (b_s z - b(t,z))\frac{V_z}{F} - \frac{1}{2}\beta^2\frac{V_z^2}{V_{zz}} + e^{-\rho(t-s)}L(z) = 0$$

- with reference to the definition of the benchmark fund.
- For a given asset allocation strategy w the performance of the fund is

$$dZ = \frac{dX}{F} - \frac{X(rF - b_s)}{F^2}dt$$

$$= wZ\big((\lambda - r)dt + \sigma dW(t)\big) + \frac{(b_s Z - b(t,Z))}{F}dt.$$

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A SUGGESTED SOLUTION

- The last two equations clearly suggest that the pension benefits are linearly linked to the performance of the fund.
- Thus, it is natural to propose as a withdrawal rule

$$b(t, Z) = b_s Z$$

- This implies that the pensioner can withdraw a fraction of the amount b_s depending on the performance of the fund relative to the benchmark fund F as measured by Z.

A POSTULATED VALUE FUNCTION

- In order to facilitate a solution the value function V is commonly postulated (cf. Björk (2009), Ch. 19) to be seperable in the varables t and z as follows:
$$V(t,z) = e^{-\rho(t-s)}H(t)L(z).$$
- The first order condition presented above now takes the form
$$w = -\frac{\beta L'}{\sigma z L''}$$

We note that our assumptions on the derivatives of the loss function give $w > 0$.

PROPERTIES OF THE OPTIMAL STRATEGY

- This last equation is independent of H.
- Thus, in general the optimal strategy depends only on
  - the properties of the loss function L as portrayed by the function's first and second derivatives,
  - the current performance of the fund as measured by z and
  - the risk of the stock as measured by η, where

$$\eta = \frac{\beta}{\sigma} = \frac{\lambda - r}{\sigma^2} .$$

INFERENCE ON Z (1)

- Substituting the optimal asset allocation into the state equation gives for fund performance
$$dZ = -\frac{L'}{L''}(\beta^2 dt + \beta dW(t)).$$
- By the Girsanov theorem the Brownian motion becomes under the risk-neutral measure Q, cf. Björk (2009, Ch. 11),
$$dW^Q(t) = dW(t) + \beta dt,$$
- where again β is the Sharpe ratio.

INFERENCE ON Z (2)

- Consequently,

$$dZ = -\frac{\beta L'}{L''} dW^Q(t)$$

- so we can infer that and Z, and hence $b(t, Z)$ by virtue of being a multiple of Z, are local martingales, i.e., have zero drift, under the risk-neutral measure Q.

FUND PERFORMANCE UNDER RISK-NEUTRAL AND OBJECTIVE MEASURES

- We have above derived that the benefit rule given by $b(t, Z) = b_s Z$ yields a fund performance that is a martingale under the risk-neutral measure.
- We now define the fair-value drawdown such that the performance of the pension fund is a martingale under the objective measure.

INCOME DRAWDOWN MAKING Z A MARITINGALE UNDER THE OBJECTIVE MEASURE

- Substituting the seperable form of V into the HJB equation we get
$$\frac{H' - \rho H + 1}{H} = \frac{1}{2}\beta^2 \frac{L'^2}{LL''} - (b_s z - b(t,z))\frac{L'}{FL}.$$
- This equation suggests income drawdown of the form $b(t,Z) = b_s Z + \phi(Z)F(t)$.
- Skipping over some details suggests setting $\phi(Z) = -\beta^2 \frac{L'}{L''}$ thus making Z a martingale under the objective measure.

IMPLICATIONS FOR ASSET ALLOCATION

- We now consider the implications for asset allocation of a simple form of the loss function L.
- If $L(z) = Be^{-\alpha z}$ where $B, \alpha > 0$ the first order condition for the HJB equation gives

$$w = -\frac{\beta L'}{\sigma z L''} = \frac{\eta}{\alpha z}$$

- so investing in the risky asset decreases with higher risk aversion. Also, if the fund is performing better than the benchmark fund it is optimal to invest a smaller share in the stock.

IMPLICATIONS FOR BENEFITS

- Next, considering the fair value drawdown in this case we obtain $\phi(Z) = -\beta^2 \frac{L'}{L''} = \frac{\beta^2}{\alpha}$ so we get $b(t, Z) = b_s Z + \frac{\beta^2}{\alpha} F(t)$.
- Thus, the fair value drawdown would allow for a higher withdrawal compared to the pure performance drawdown considered above.
- The additional term $\frac{\beta^2}{\alpha} F(t)$ can be considered being an additional risk premium.

CONCLUSIONS

- We have explored a fairly general stochastic model of a pension fund.
- This model gives rise to conclucions on asset allocation and links between sustainable pension benefits and fund size and fund performance.
- Given the relatively unrestrictive assumptions we conclude that increased confidence can be placed on the validity of outcomes regarding asset allocation and pension benefits.

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