The importance of being negative
An invitation to nonfinite axiomatizability results

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Why This Talk at This Event?

My Tenet

(Theoretical) Computer Science and its forefather Mathematical Logic are pretty much unique in their development of mathematical methodology for proving negative results.

Christos Papadimitriou's Viewpoint

Negative results are the only possible self-contained theoretical results in Computer Science.
Successful exploratory theoretical research is bound to produce predominantly negative results. (From "Database metatheory: Asking the big queries")

What kind of negative results is this talk about?

The Role of Equalities Between Programs

Motto: In Computer Science, we use formal languages to communicate with machines and describe expected properties of computations.

Fact of Life: We often need to know when two syntactically different descriptions are describing the "same thing". Examples?

- Optimization in compilers.
- Program analysis/partial evaluation...
- Correctness: Is SPECification equivalent to IMPlementation?
- Specification of abstract data types.

Tenet: Equational logic can be used to capture "valid" equivalences.

Formal languages are algebras!

Syntax = Signature: A collection Σ of operations to construct complex expressions from simpler ones that we have already built.

Example: The operations of the max-plus algebra

Let V be a countably infinite set of variables.

1. 0 and each $x \in V$ are terms.
2. If t and u are terms, then so are $t \vee u$ and $t + u$.

For example, x, $x \vee 0$, $x \vee x$ and $(x \vee y) + z$ are terms.

The set of all terms over a signature Σ form the so-called term algebra.

The Max-plus algebra of the natural numbers

Semantics: A natural algebra for interpreting max-plus terms

$(\mathbb{N}, \vee, +, 0)$, where $+$ is ordinary addition and $\vee$ stands for the maximum between two numbers.

Question: How can we express properties of this algebraic structure?

Use equational logic!

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Equations and their validity

An equation is a formula of the form $t = u$, where t and u are terms.

Examples of equations over the signature for max-plus terms

$$
\begin{aligned}
x \vee 0 &= x \\
x \vee x &= x \\
(x \vee y) + z &= (x + z) \vee (y + z) \\
(x + x) \vee (y + y) &= (x \vee y) + (x \vee y).
\end{aligned}
$$

These equations are all valid/sound in the algebra $(\mathbb{N}, \vee, +, 0)$.

Note: In Universal Algebra, valid equations state properties of an algebra. They are not meant to be solved!

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Finite, Complete Axiomatizations

Question: When does one "really understand" an algebra equationally?

The Challenge

Given some algebraic signature Σ, and some congruence $\sim$ over terms

Is there a finite set $\mathcal{E}$ of Σ-equations $s = t$ such that

$$t \sim u \quad \Leftrightarrow \quad \mathcal{E} \vdash t = u$$

for all (closed) Σ-terms t, u?

$\mathcal{E}$ is called a sound and complete axiomatization. If $\mathcal{E}$ is finite, then the algebra is finitely based.

Why is This an Interesting Game (for a Computer Scientist)?

Answer 1

The axiomatic method is a very powerful method of scientific analysis, so studying its power in Computer Science must be interesting!. . . And I like it!

Answer 2

An equational axiomatization

1. tells ye all ye need to know about your notion of program equivalence;
2. allows you to relate it to other types of program equivalence by simply looking at laws;
3. may form the basis for program verification tools based on theorem proving technology.

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The Cold Shower

Main General "Technical" Message of the Talk

The life of a computer scientist or a mathematician is equationally hard.

In many situations, the collection of valid equivalences cannot be "captured" by means of a finite collection of equations. This holds true even for very simple languages and algebras!

Rest of the Talk

Examples of negative results of that type, hinting at how they are proved.

And now for a little technical content!

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A “Simple” Algebra

A Question Equivalent to a Process Algebraic One

Is the algebra $(\mathbb{N}, \vee, +, 0)$ finitely based?

Theorem (Ésik, Ingolfsdottir and yours truly): No!

1. For each $n \geq 1$, the collection of valid equations in at most n variables is not complete.
2. Moreover, no finite collection of valid equations can prove all of

$$nx \vee ny = n(x \vee y) \quad (n \geq 1) \ .$$

So the collection of valid equations in two variables is not finitely based.

How do we prove these results?

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The Proof Strategy I

Aim: Prove that, for each $n \geq 1$, the collection of valid equations in at most n variables is not complete.

Recipe (for three people): Step 1

Find an infinite family of “hard valid equations”. In our case,

$$t_n \vee u_n \quad = \quad u_n \qquad (n \geq 1)$$

where

$$\begin{aligned} t_n &= x_1 + x_2 + \cdots + x_{n-1} + x_n \\ u_n &= (2x_1 + x_3 + \cdots + x_{n-1} + x_n) \vee \\ &\quad (x_1 + 2x_2 + x_4 + \cdots + x_{n-1} + x_n) \vee \\ &\quad \vdots \\ &\quad (x_2 + x_3 + x_4 + \cdots + x_{n-1} + 2x_n) \end{aligned}$$

The Proof Strategy II

Recipe (for three people): Step 2

For each $n \geq 1$, find a model for the all the valid equations in at most n variables in which $t_{n+1} \vee u_{n+1} = u_{n+1}$ fails.
Parental advisory warning: This is not easy!

The result now follows from Birkhoff's completeness theorem for equational logic.

Theorem of possible interest for you

The algebras $(\mathbb{N}, \vee, +, 0)$, $(\mathbb{Q}_{\geq 0}, \vee, +, 0)$ and $(\mathbb{R}_{\geq 0}, \vee, +, 0)$ have the same equational theory.

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A Menagerie of Nasty Algebras

Question

Is the max-plus algebra the only “simple” non-finitely axiomatizable one over the natural numbers?

No way!

The bestiary of “nasty algebras” includes

- $(\mathbb{N}, \wedge, +, 0)$,
- various tropical semirings, and
- the algebra in Tarski's high school algebra problem.

The life of a concurrency theorist/mathematician is equationally hard, indeed. . .

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A Pearl of Wisdom from John D. Barrow

By the end of our journey, I hope that readers will have come to see that there is more to impossibility than first meets the eye. Its role in our understanding of things is far from negative. Indeed, I believe that we will gradually come to appreciate that the things that cannot be known, that cannot be done, and cannot be seen, define our Universe more clearly, more completely, and more sharply than those that can. (From *Impossibility: The Limits of Science and the Science of Limits*)

Thank You!

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Any Questions?

Shameless PR

Interested in our work/collaborating with Anna and me?

- Read the papers by Anna and me available from my publication page. (Search words: Axiomatizing, nonfinitely based, nonexistence, ...)
- Come and talk to us.
- Take part in ICE-TCS events. Drop me a line at `luca@ru.is` to be added to our mailing list.

Appendix: Techniques for Proving Negative Results

Technical Problem

How can one prove negative results?

1. Compactness theorem: Find an (infinite) sound and complete axiomatization for which no finite subset is complete.
2. Model-theoretic approach: For each finite sound axiomatization $\mathcal{E}$, find a model for $\mathcal{E}$, and a sound equation that fails in this model.
3. Proof-theoretic approach: For each finite sound axiomatization $\mathcal{E}$, find a property of equations that
   (A) is satisfied by all instantiations of axioms in $\mathcal{E}$,
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