Wilf Classification of Mesh Patterns of Short Length

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Permutations

Definition

Let A be a finite, non-empty set. A one-to-one correspondence from A to itself is a permutation. Let $A = \{1, 2, \ldots, n\}$ and denote a permutation as a word, $\pi = \pi_1\pi_2\ldots\pi_n$. Let S_n be the set of all permutations of length n.

Example

The word $\pi = 1324$ is a permutation of the set $\{1, 2, 3, 4\}$ with $\pi_1 = 1$, $\pi_2 = 3$, $\pi_3 = 2$ and $\pi_4 = 4$.

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Classical patterns

Definition

A classical pattern is a permutation in S_k.

Example

The pattern 231 can be drawn as follows, where the horizontal lines represent the values and the vertical ones denote the positions in the pattern.

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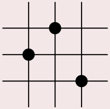
231 = 

Occurrence/avoidance of patterns

Definition

We say that a pattern occurs in a permutation if there is a subsequence whose letters are in the same relative order of size as the letters of the pattern. If a pattern does not occur in a permutation, the permutation avoids the pattern.

Example

The permutation 362451 contains the pattern 231 =

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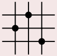
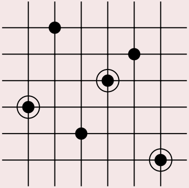
The permutation 362451 contains the pattern 231 = 



Vincular patterns

Definition

For a vincular pattern, also called generalized patterns, to occur in a permutation, the pattern may require letters to be adjacent in the permutation.

Example

The permutation on the left, 4**26351**, contains the vincular pattern 1-23 = and the permutation on the right, 6**34251**, avoids the pattern

Vincular patterns were first defined by Babson and Steingrímsson in 2000

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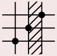
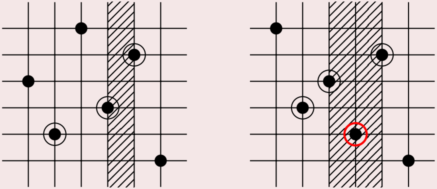
The permutation on the left, 4**2**6**35**1, contains the vincular pattern 1-23 =  and the permutation on the right, 6**3425**1, avoids the pattern



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Bivincular patterns

Definition

A bivincular pattern is a pattern that can put constraints on positions and values in a permutation.

Example

The permutation on the left, **243**6**1**5, contains the pattern $\overline{1-2}3 =$ and the permutation on the right, **435**2**6**1, avoids the pattern

Bivincular patterns were first introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev in 2010

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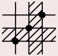
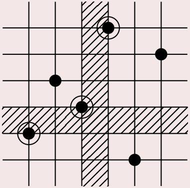
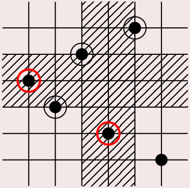
The permutation on the left, **2436**15, contains the pattern $\overline{1-2}3 =$  and the permutation on the right, **43526**1, avoids the pattern





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Mesh patterns

Definition

A mesh pattern is a pair (τ, R), where τ is a permutation in S_k and R is a subset of $[\![0, k]\!] \times [\![0, k]\!]$.

Example

The permutation on the left, **164235**, contains the pattern
and the permutation on the right, **16534**2, avoids the pattern

Mesh patterns were first introduced by Brändén and Claesson in 2010

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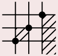
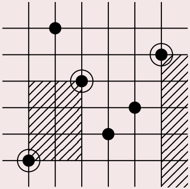
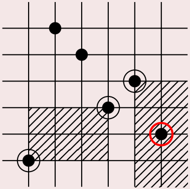
Example

The permutation on the left, **1**6**4**23**5**, contains the pattern  and the permutation on the right, **1**65**34**2, avoids the pattern





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Definition

Recall that two patterns p and q are Wilf-equivalent if equally many permutations of length n avoid p and q, for all n.

Wilf-equivalence is one of the big questions in the study of patterns. Our goal was to start the Wilf-classification of mesh patterns.

- Simion, Schmidt
  Restricted permutations 1985
- Babson, Steingrímsson
  Generalized permutation patterns and a classification of the Mahonian statistics 2000
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- We have studied patterns of length 2
- The number of mesh patterns of length 2 is 1024
- We used the mathematics software system Sage to help us sort the patterns by Wilf-equivalence
  - Reverse
  - Complement
  - Inverse
  - Toric shift
  - Up-shift
  - The Shading Lemma
- Using these operations we find that there are at most 65 Wilf-classes for mesh patterns of length 2.
- Robert Parviainen had already Wilf-classified bivincular patterns of length 2 (and 3), and thus there are at most 58 Wilf-classes left to prove.

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Definition

Let $[i, j]$ denote the box whose corners have coordinates $(i, j), (i, j+1), (i+1, j+1)$ and $(i+1, j)$.

Lemma (The Shading Lemma)

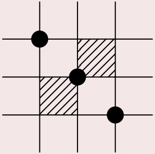
Let (τ, R) be a mesh pattern of length n such that $\tau(i) = j$ and $[i, j] \notin R$. If all of the following conditions are satisfied:

- *The box $[i-1, j-1]$ is not in R;*
- *At most one of the boxes $[i, j-1]$, $[i-1, j]$ is in R;*
- *If the box $[\ell, j-1]$ is in R ($\ell \neq i-1, i$) then the box $[\ell, j]$ is also in R;*
- *If the box $[i-1, \ell]$ is in R ($\ell \neq j-1, j$) then the box $[i, \ell]$ is also in R;*

then the patterns (τ, R) and $(\tau, R \cup \{[i, j]\})$ are equivalent. Analogous conditions determine if the other neighboring boxes can be added to R.

Example

The following equivalence is found by using the Shading Lemma



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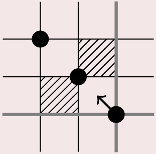

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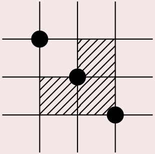

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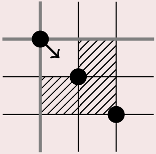

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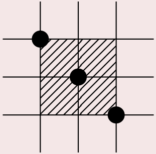

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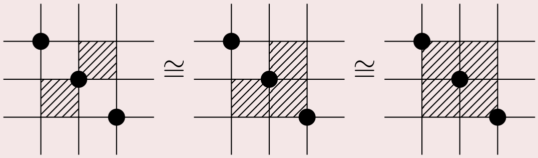

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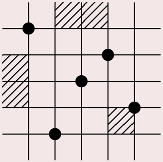

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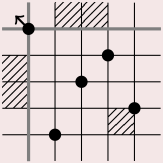

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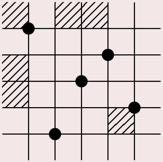

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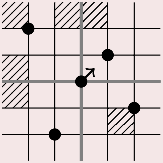

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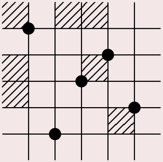

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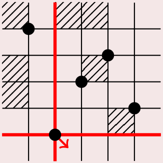

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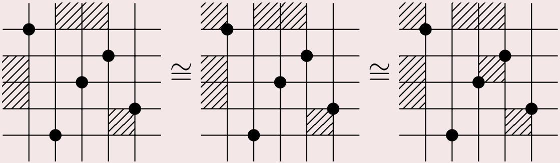
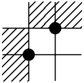


Representative	Formula	# of patterns
	1	28
	$(n-1)!$	40
	$a_n = n \cdot a_{n-1} - a_{n-2}$	32
	$a_n = (n-1)a_{n-1} + (n-2)a_{n-2}$	32
	$[x^n]\left(1 - \frac{1}{\sum_n n! x^n}\right)$	4
	$\sum_{i=1}^{n-1} \frac{(n-1)!}{i}$	84
	$\left[\frac{x^n}{n}\right] \log\left(1 + \sum_{i=1}^{n} (i-1)! \cdot x^i\right)$	60

Representative	Formula	# of patterns
	$n! - \sum_{i=1}^{n-1} \sum_{\ell=1}^{i} (i-\ell)!(n-i-\ell)!\ell!$	4
	$n! - \sum_{k=0}^{n-2} \sum_{j=0}^{k} j!(k-j)!(n-2-k)!$	4
	$n! - \sum_{i=0}^{n-2} i!(n-1-i)!$	16
	$n! - \sum_{k=1}^{n-1} (k-1)!(n-k-1)!$	24
	$n! - (n-1)! + [x^n]\frac{F(x)}{1+xF(x)}$	8
	where $F(x) = \sum_{n\geq 0} n!x^n$	

There is exactly one permutation of length n avoiding the following pattern



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Example

The permutation 624153 contains the pattern

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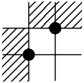

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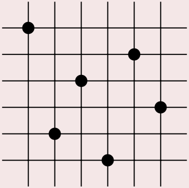

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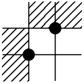

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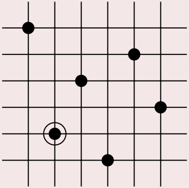

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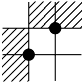

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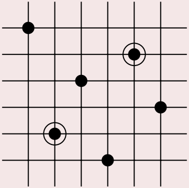

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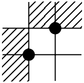

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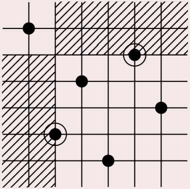
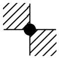


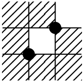
Definition

A fixed point $\pi(k) = k$ is called a strong fixed point if we have $\pi(i) < k$ for $i < k$ and $\pi(j) > k$ for $j > k$.

A strong fixed point in a permutation π is one that is both a left-to-right maximum and a right-to-left minimum. This is the same as π containing the pattern



Permutations containing the pattern



are those starting with the letter 1 and have a strong fixed point. Therefore the number of permutations that avoid the pattern is

$$n! - (n-1)! + [x^n]\frac{F(x)}{1 + xF(x)}$$

where $F(x) = \sum_{n \geq 1} (n-1)! x^{n-1}$.

At this point we have found formulas for the number of permutations avoiding 776 patterns out of the 1024 we started with.

Outcome

- Intro article about pattern avoidance in Verpill
- New translations added to the Icelandic mathematical dictionary
- Code for Sage
- Accepted to Permutation Patterns 2011, a conference in California June 20-24

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Thank you!

Any questions?